On the Nature of Infrared Singularities in $d \leq 2$ Disordered Interacting Systems.

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We address the problem of infrared singularities in the perturbation theory for disordered interacting systems in $d \leq 2$. We show that a typical, sufficiently large interacting system exhibits a linear instability in the spin triplet channel. In the density–density channel, although stability is preserved, a large number of soft modes is accumulated. These phenomena are responsible for the instability of the weak–interacting fixed point. Although generic, the unstable direction and soft modes are highly sample specific and can not be effectively captured by conventional techniques based on an averaging procedure. Rather, the instability is determined by the largest eigenvalues of the polarization operator. We propose to employ the optimal fluctuation method for evaluating the probability of such events.

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The physics of two–dimensional ($d = 2$) disordered electronic systems has received a new impetus since the accumulation of experimental evidence of a metal–insulator transition in zero magnetic field [1]. Other remarkable experimental developments are associated with $d = 2$ finite size structures – quantum dots [2]. The data indicate the crucial role of e–e interactions in these disordered (chaotic) systems. We also point out that in both cases spin degrees of freedom seem to play a major role.

On the theoretical side it was known since the work of Altshuler and Aronov [3] that various physical observables acquire infrared singular corrections in the presence of e–e interactions. Using a finite system size, L, as a cutoff [4], one may write the correction to an observable, χ (which may be specific heat, conductivity, spin polarization etc.) as

$$\frac{\delta\chi}{\chi} \propto \frac{L^{2-d}}{\nu_d D}\,, \tag{1}$$

where ν_d is a d–dimensional density of states at a Fermi level and D is a diffusion constant. Hereafter, L^{2-d} should be understood as $\ln L/l$ with l being a microscopic length (mean free path [5]) for the $d = 2$ case. Once the system size becomes comparable with the characteristic length $\xi_d \equiv (\nu_d D)^{1/(2-d)}$ ($\xi_2 = l e^{\nu_2 D}$), the validity of the first order perturbation theory breaks down, and expression (1) can not be applied. Despite the fact that ξ_d has an interpretation as the localization length, Eq. (1) has nothing to do with the actual Anderson localization. In fact, in the presence of a small magnetic field the physical localization length may be much larger than ξ_d. Thus, the behavior of an observable χ at $L > \xi_d$ may be studied quite independently of the onset of the Anderson localization.

Finkel'stein [6,7] has developed a renormalization group approach to the problem. His results show that a weak–disorder and weak–interaction fixed point is unstable at $d \leq 2$. As a result, the large scale behavior of the system is governed by some other fixed point. He has also noticed that the system acquires a tendency towards a partial spin polarization and pointed out that the spin susceptibility diverges independently of all (and before) other quantities [7]. In spite of considerable further efforts in this direction [8–10], the physical nature of the new fixed point remains to a large extent unclear. The ultimate fate of the corrections, Eq. (1), in the region $L > \xi_d$ is also unknown.

The purpose of this letter is to shed some light on the above questions. We concentrate primarily on the spin triplet interaction channel, restricting ourselves to few remarks concerning spin singlet (density–density) channel. We argue that for each *particular realization* of disorder and $L > \xi_d$ the system exhibits a *linear response* instability in the spin triplet channel. On the mean–field level this instability leads to the spontaneous breaking of the spin–rotation symmetry and partial spin polarization. The questions concerning stability of true long-range order with respect to fluctuations in the thermodynamic limit are well beyond the scope of this letter. For a finite size system at sufficiently low temperatures, however, the fluctuations will not destroy spontaneous magnetization. The existence of an unstable direction (in the functional space of possible spin polarizations) is a generic property of any disordered system. Its concrete shape, however, depends on the polarization operator for a given disorder realization and is extremely sample dependent. Methods which employ an ensemble averaging procedure at an early stage of the calculations are bound to loose information about this instability since they use the disorder-averaged polarization operator. Instead of being linear, the effect appears to be encoded into higher non–linear corrections. The stability of the paramagnetic state for a given system has very little to do with the averaged susceptibility. The importance of fluctuations of various susceptibilities in disordered metals was stressed by several authors [11,12] (see also [13] for applications to superconductors). In fact, the stability of the paramagnetic state is determined by atypically large eigenvalues

of the polarization operator [12]. Thus, methods treating the optimal fluctuations seem to be more adequate. These methods were initially developed for the problem of "Lifshitz tails" by Zittartz and Langer [14] and used recently in the problem of pre–localized states [15]. Below we adopt them for the treatment of interacting electrons.

Consider a gas of interacting electrons, moving in a random potential. In terms of the slow degrees of freedom the interaction Hamiltonian takes the form

$$H = \frac{1}{2\nu_d}\iint d^d r\, d^d r' \Big[V_s(r-r') n(r) n(r') - \hat{V}_t \mathbf{s}(r)\mathbf{s}(r) \Big] , \tag{2}$$

where the charge, $n(r) = \sum_\sigma \psi^\dagger_\sigma(r)\psi_\sigma(r)$, and spin, $\mathbf{s}(r) = \sum_{\sigma\sigma'} \psi^\dagger_\sigma(r)\sigma_{\sigma\sigma'}\psi_{\sigma'}(r)$, densities are slowly varying on the scale λ_F; $\hat{V}_t = V_t\delta(r-r')$. We write the partition function as an imaginary time functional integral [16] and perform the Hubbard–Stratonovich transformation of the singlet and triplet interaction terms by means of fields Φ and $\mathbf{H}$ correspondingly. Having in mind developing a Landau–Ginzburg mean–field theory at finite temperature T, we restrict ourselves to the zero Matsubara frequency only, $\mathbf{H}(r) = \mathbf{H}(r, \omega_m = 0)$. The subsequent integration over the fermionic degrees of freedom results in $\det[1 + G(i\Phi + \mathbf{H}\sigma)]$, which may be now expanded in powers of Φ and $\mathbf{H}$. In this way one obtains

$$\int D\mathbf{H} \exp\left\{ -\frac{\nu_d T}{2} \iint \mathbf{H}(r) \Big[\hat{V}_t^{-1} - \hat{\Pi}(r,r') \Big] \mathbf{H}(r') + \hat{\Gamma}\mathbf{H}^4 \right\} , \tag{3}$$

where $\hat{\Gamma}\mathbf{H}^4$ designate non–linear terms. The central quantity of interest is the static polarization operator (PO), defined as

$$\hat{\Pi}(r,r') = -\nu_d^{-1} T \sum_{\epsilon_n} G_{\epsilon_n}(r,r') G_{\epsilon_n}(r',r) , \tag{4}$$

where $G_{\epsilon_n}(r,r')$ is a *sample specific* Green function and $\epsilon_n = \pi T(2n+1)$. The Gaussian part of the functional integral, Eq. (3), becomes unstable if the positively defined Hermitian operator $\hat{\Pi}(r,r')$ has at least one eigenvalue larger than V_t^{-1}. We are faced, thus, with the spectral problem for the PO:

$$\int d^d r'\, \hat{\Pi}(r,r') \Phi_n(r') = \lambda_n \Phi_n(r) ; \tag{5}$$

$\int |\Phi|^2 d^d r = 1$. Specifically, we are interested in the large λ tail of the PO spectral density since the spontaneous symmetry breaking will be determined by the largest eigenvalue. The spectral density is defined as

$$N(\lambda) \equiv \left\langle \sum_n \delta(\lambda - \lambda_n) \right\rangle , \tag{6}$$

where the angular brackets denote disorder averaging. Employing the optimal fluctuation we shall demonstrate that

$$N\left(\lambda > \lambda^{\rm typical}\right) \propto \exp\left\{ -f\left(\lambda \frac{\nu_d D}{L^{2-d}} \right) \right\} , \tag{7}$$

with $f(x)$ being a certain universal function ($\sim x^2$ – in a simplest approximation). Eq. (7) demonstrates that the condition for the appearance of a large ($\lambda > V_t^{-1}$) eigenvalue is $L^{2-d} > \nu_d D/V_t$ which coincides precisely with the breakdown of the perturbation theory, Eq. (1). It is clear now from Eq. (3) that once such an eigenvalue is formed, the system develops a non–trivial saddle point with $\mathbf{H}(r) \neq 0$, which describes a spin polarized state. A new expansion arround this sample–specific minimum should be developed.

The (replicated) ensemble averaged version of the effective action in Eq. (3) is

$$\frac{\nu_d T}{2} \iint \mathbf{H} \Big[\hat{V}_t^{-1} - \langle \hat{\Pi} \rangle \Big] \mathbf{H} + \left(\langle \hat{\Gamma} \rangle - \frac{1}{4} \langle\langle \hat{\Pi}^2 \rangle\rangle \right) \mathbf{H}^4 + \dots . \tag{8}$$

The average PO, $\langle \Pi(r,r') \rangle = \delta(r-r')$, is an operator with all eigenvalues equal to unity and thus with no tails in spectral density [17]. This fact ensures the stability of the Gaussian integral for $V_t^{-1} > 1$, which is the Stoner criterion for the ferromagnetic instability [18]. The information about anomalously large eigenvalues of $\hat{\Pi}$ is now hidden in the higher order non–linear terms. Thus the basic fact about the presence of a linear instability in the theory appears to be obscured by the averaging procedure.

Let us make a few remarks concerning the singlet interaction channel. The corresponding functional integral over a scalar field Φ has a Gaussian part of the form

$$\frac{\nu_d T}{2} \iint d^d r\, d^d r' \Phi(r) \Big[V_s^{-1}(r-r') + \hat{\Pi}(r,r') \Big] \Phi(r') . \tag{9}$$

Note the plus sign in front of PO! The kernel is now strictly positively defined and can not exhibit an instability. It may have, however, very soft (and long-wavelength) modes associated with a *small* λ ($\lambda \geq 0$) tail of the spectral density $N(\lambda)$ [19]. These atypically soft modes may lead to singularities in the singlet channel. One should be careful with these arguments, however, since there is no reason to neglect non–zero Matsubara frequencies in this situation. For $\omega_m > DQ^2$ even the average PO has small eigenvalues. The fact that the divergence of singlet and triplet quantities is associated with the opposite tails of $N(\lambda)$ may be related to Finkel'stein's results quoted above.

We turn now to the evaluation of the tails of the PO spectral density, $N(\lambda)$. To this end we define the corresponding Green function operator as

$$\hat{\mathcal{G}}_\lambda(r,r') = (\hat{\lambda} - i\eta - \hat{\Pi})^{-1} = \sum_n \frac{\Phi_n(r)\Phi_n^*(r')}{\lambda - i\eta - \lambda_n} \qquad (10)$$

and $N(\lambda) = \pi^{-1}\Im\mathrm{Tr}\langle\hat{\mathcal{G}}_\lambda\rangle$. To perform disorder averaging we need to know the statistics of the random operator $\hat{\Pi}(r,r')$. A straightforward diagrammatic calculation of its second cumulant yields at $T=0$

$$Q_{ss'}^{rr'} \equiv \langle\langle\hat{\Pi}(r,r')\hat{\Pi}(s',s)\rangle\rangle = \qquad (11)$$
$$\frac{1}{2\pi^2\nu_d^2}\int_0^\infty \varepsilon d\varepsilon D_\varepsilon(r-s')D_\varepsilon(s'-r')D_\varepsilon(r'-s)D_\varepsilon(s-r)\,.$$

Here $D_\varepsilon(r)$ is a propagator of the classical diffusion operator

$$D_\varepsilon(r) = \sum_Q \frac{e^{iQr}}{DQ^2+|\varepsilon|} = \frac{L_\varepsilon^{2-d}}{D}F_d\left(\frac{r}{L_\varepsilon}\right), \qquad (12)$$

where $L_\varepsilon = \sqrt{D/\varepsilon}$ and $F_d(x)$ is a dimensionless function with the assimptotic behavior $F_d(0) = \mathrm{const}$ and $F_d(x\to\infty)\sim e^{-x}$. Performing the energy integration in Eq. (11) one obtains

$$Q_{ss'}^{rr'} \approx \frac{(\nu_d D)^{-2}}{\left(|r-s'|+|s'-r'|+|r'-s|+|s-r|\right)^{4(d-1)}}\,. \qquad (13)$$

We can perform now the averaging of the Green function, $\hat{\mathcal{G}}_\lambda$, assuming the Gaussian distribution for $\hat{\Pi}$. This is certainly not an exact procedure because the higher order cumulants of $\hat{\Pi}$ are not negligible in the most interesting parameter region $L > \xi_d$. We shall follow, however, this idea since it allows us to illustrate the method and provides a basis for further generalizations. We thus obtain

$$\left\langle\hat{\mathcal{G}}_\lambda(R,R')\right\rangle = \int D\hat{\Pi}\, e^{-\frac{1}{2}\hat{\Pi}_{rr'}(Q^{-1})_{ss'}^{rr'}\hat{\Pi}^{s's}+\ln\hat{\mathcal{G}}_\lambda(R,R')}\,, \qquad (14)$$

where integration over repeated coordinates is implicit. The average value of the PO is omitted in this expression, since it leads to a trivial redefinition $\lambda\to\lambda-1$ only. We argue now, following Zittartz and Langer [14], that large λ tails are determined by the saddle point of this functional integral. Variation over $\hat{\Pi}_{rr'}$ leads to the equation for the optimal realization of the PO

$$\bar{\Pi}(r,r') = Q_{ss'}^{rr'}\frac{\hat{\mathcal{G}}_\lambda(R,s)\hat{\mathcal{G}}_\lambda(s',R')}{\hat{\mathcal{G}}_\lambda(R,R')}\,. \qquad (15)$$

In the close vicinity of some atypically large eigenvalue, λ_0, the Green function may be well approximated by the single term in the sum, Eq. (10), $\hat{\mathcal{G}}_\lambda(r,r') = \Phi_0(r)\Phi_0^*(r')/(\lambda-\lambda_0)$, leading to

$$\bar{\Pi}(r,r') = \frac{1}{\lambda-\lambda_0}\,Q_{ss'}^{rr'}\Phi_0^*(s)\Phi_0(s')\,. \qquad (16)$$

Again the d–dimensional integration is assumed over repeated coordinates. Finally, substitution of Eq. (16) into Eq. (5) results in the nonlinear self-consistency equation for the typical eigenfunction corresponding to a large eigenvalue, λ_0,

$$\lambda_0\Phi_0(r) = \frac{1}{\lambda-\lambda_0}\,Q_{ss'}^{rr'}\Phi_0(r')\Phi_0^*(s)\Phi_0(s')\,. \qquad (17)$$

To avoid the solution of this non–linear integral equation one may use purely dimensional arguments. To this end let us write this equation in terms of dimensionless coordinates $x = r/L$ and dimensionless eigenfunctions $\phi_0(x) = L^{d/2}\Phi_0(r)$. We also employ Eq. (13) for the cumulant Q:

$$\phi_0(x) = \left[\frac{L^{2-d}}{\nu_d D\sqrt{\lambda_0(\lambda-\lambda_0)}}\right]^2 \times \qquad (18)$$
$$\int_0^1 \frac{d^dx'\,d^dy\,d^dy'\,\phi_0(x')\phi_0^*(y)\phi_0(y')}{\left(|x-y'|+|y'-x'|+|x'-y|+|y-x|\right)^{4(d-1)}}\,.$$

To ensure the existence of a normalizable $\left(\int_0^1|\phi_0|^2d^dx = 1\right)$ solution of this equation one must require that the expression in the square brackets on its r.h.s. is of order unity. We conclude that

$$\lambda-\lambda_0 \approx \lambda_0^{-1}\left(\frac{L^{2-d}}{\nu_d D}\right)^2\,. \qquad (19)$$

The last step is to find the statistical weight of the optimal realization $\bar{\Pi}$, which has λ_0 as an eigenvalue. Its statistical weight is obviously given by $\bar{\Pi}Q^{-1}\bar{\Pi} = \lambda_0/(\lambda-\lambda_0)$, where we employed Eqs. (16), (17) and normalization condition for Φ_0. The last expression together with Eq. (19) implies that

$$N(\lambda_0) \propto \exp\left\{-c_d\left(\lambda_0\frac{\nu_d D}{L^{2-d}}\right)^2\right\}\,, \qquad (20)$$

where c_d is a numerical constant of order unity. The fact that Gaussian tails are obtained may be traced back to the assumption about the Gaussian distribution of the PO and should not be taken too seriously in the regime $L > \xi_d$. However, the dependence of the spectral density on the parameter $\lambda\nu_d D/L^{2-d}$ follows from pure dimensional analysis and may be justified for more realistic assumptions concerning PO statistics. In fact, the Gaussian spectral density, Eq. (20), is the most pessimistic estimate for the tails. Indeed, all the effects we have neglected (higher order cumulants of $\hat{\Pi}$, and onset of Anderson localization) should increase fluctuations of the PO, making the tails decrease slower than Gaussian.

It remains a challenging problem to calculate the spectral density of the PO under more realistic assumptions. The "pessimistic" result, Eq. (20), is already sufficient to demonstrate our main point: one may find an arbitrarily large eigenvalue of the PO if the system size is taken to be large enough. This leads to a ferromagnetic instability in the triplet channel for arbitrarily small interaction, V_t.

In the most interesting $d = 2$ case the argument in Eq. (20) has the form $\lambda_0 g(\ln L/l)^{-1}$, where $g = \nu_2 D > 1$ is a dimensionless conductance of a $2d$ metal. To find an eigenvalue of the PO $\lambda_0 > V_t^{-1} > 1$ one should typically consider a sample with a size, $L > l \exp\{g/V_t\}$. This still may be much smaller than the localization length in the unitary ensemble, which is of the order $l \exp\{g^2\}$. This is to say that, contrary to a naive interpretation of Eq. (20), one may encounter the ferromagnetic instability well inside the metallic side of the Anderson transition.

We want to point out that in low dimensional ($d \leq 2$) systems the spontaneous symmetry breaking occures into an *extended* state, rather than into a localized one (as it happens in 3d systems [12]). This is clear from the extended nature the solution of Eq. (18). In this case the role of nonlinear terms ($\Gamma \mathbf{H}^4$ in Eq. (3)) reduces to merely determining the amplitude of the spontaneous globally coherent magnetization.

The aim of this letter is to elucidate the source of some problems in the theory of $d \leq 2$ disordered interacting electrons. The traditional perturbation theory and RG treatment lead to infrared divergence and instability of the weak–coupling fixed point correspondingly. They do not provide a simple physical reason for the singularities, nor do they explain the nature of the strong coupling fixed point. We argue that at least some of the difficulties can be traced to (i) the ferromagnetic instability of a given system in the triplet channel, and (ii) to the accumulation of many soft modes in the singlet channel. The latter necessitates taking into account non–linear terms in the action. Both the unstable directions and the soft modes, although generic, are sample specific and do not survive (traditional) ensemble averaging procedure. It would be desirable to construct a theory, which first adjusts the integration directions (bosonic fields) to the concrete sample configuration and only then performs the averaging.

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[1] S. V. Kravchenko *et al.*, Phys. Rev. **B 63**, 8039 (1994); D. Simonian *et al.*, Phys. Rev. **B 55**, 13421 (1997); S. V. Kravchenko *et al.*, cond-mat/9709255; D. Popovic, A. B. Fowler, and S. Washburn, Phys. Rev. Lett. **79**, 1543 (1997); Y. Hanein *et al.*, cond-mat/9709184.

[2] J. A. Folk *et al.*, Phys. Rev. Lett. **76**, 1699 (1996); S. R. Patel *et al.*, cond-mat/9708090.

[3] B. L. Altshuler, and A. G. Aronov, Zh. Eksp. Teor. Fiz. **77**, 2028 (1979) [Sov. Phys. JETP **50**, 968 (1979)]; B. L. Altshuler and A. G. Aronov, in *Electron–Electron Interaction In Disordered Systems*, ed. A. J. Efros, and M. Pollak, pp. 1–153, Elsevier Science Publishers, North–Holand, 1985.

[4] Alternately one may think about infinite system and use a temperature length, $L_T \equiv \sqrt{D/T}$, as cutoff.

[5] As pointed out by A. M. Rudin, I. L. Aleiner, and L. I. Glazman, Phys. Rev. **B 55**, 9322 (1997), the actual ultraviolet cutoff in the logarithm should rather be λ_F than l.

[6] A. M. Finkel'stein, Zh. Eksp. Teor. Fiz. **84**, 168 (1983) [Sov. Phys. JETP **57**, 97 (1983)]; A. M. Finkel'stein, *Electron Liquid in Disordered Conductors*, v. 14 of *Soviet scientific reviews*, ed. I. M. Khalatnikov, Harwood Academic Publishers GmbH, London, 1990;

[7] A. M. Finkel'stein, Z. Phys. B, **56**, 189 (1984).

[8] D. Belitz, and T. R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).

[9] C. Castellani, and C. Di Castro, Phys. Rev. **B 34**, 5935 (1986).

[10] S. Sachdev, N. Read, and R. Oppermann, Phys. Rev. **B 52**, 10286 (1995).

[11] A. Zyuzin, B. Spivak, JETP Lett. **43**, 234 (1986); B. Spivak, A. Zyuzin, in *Mesoscopic Phenomena in Solids*, Eds. B. L. Altshuler, P. A. Lee and R. A. Webb p. 37, Elsevier Science Publishers, North–Holand, 1991.

[12] M. Milovanović, S. Sachdev and R. N. Bhatt, Phys. Rev. Lett. **63**, 82 (1989).

[13] B. Spivak and F. Zhou, Phys. Rev. Lett. **74**, 2800 (1995).

[14] J. Zittartz, and J. S. Langer, Phys. Rev. **148**, 741 (1966).

[15] B. A. Muzykantskii, and D. E. Khmel'nitskii, Phys. Rev. **B 51**, 5480, (1995); V. I. Fal'ko, K. B. Efetov, J. Math. Phys. **37**, 4935 (1997); I. E. Smolyarenko, B. L. Altshuler, Phys. Rev. **B55**, 10451 (1997).

[16] J. W. Negele, and H. Orland, *Quantum Many–Particle Systems*, Addison-Wesley Pub. Co., 1988.

[17] In fact, $\langle\hat{\Pi}\rangle(r - r')$ has smaller eigenvalues associated with eigenfunctions oscillating at $2k_F$, which are not important in the present context.

[18] S. Doniach, and E. H. Sondheimer, *Green's functions for solid state physicists*, Reading, Mass., W. A. Benjamin, 1974.

[19] One should actually be interested in small eigenvalues of the operator $V_s^{-1} + \hat{\Pi}$. In the case of the Coulomb interaction, this is similar to searching for *long-wavelength* soft modes of $\hat{\Pi}$.